LIFESPAN THEOREM FOR CONSTRAINED SURFACE DIFFUSION FLOWS

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Abstract. We consider closed immersed hypersurfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ evolving by a class of constrained surface diffusion flows. Our result, similar to earlier results for the Willmore flow, gives both a positive lower bound on the time for which a smooth solution exists, and a small upper bound on a power of the total curvature during this time. By phrasing the theorem in terms of the concentration of curvature in the initial surface, our result holds for very general initial data and has applications to further development in asymptotic analysis for these flows.

1. Introduction.

Let $f : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a family of compact immersed hypersurfaces $f(\cdot, t) = f_t : M \to f_t(M) = M_t$ with associated Laplace-Beltrami operator Δ, unit normal vector field ν, and mean curvature function H. The surface diffusion flow

$$\frac{\partial}{\partial t} f = (\Delta H)\nu, \tag{SD}$$

and the more general constrained surface diffusion flows

$$\frac{\partial}{\partial t} f = (\Delta H + h)\nu, \tag{CSD}$$

where $h : I \to \mathbb{R}$ and $I \supset [0, T)$, are the chief objects of interest for this paper. Our aim is to begin a systematic study of the regularity of the flows (CSD). We are motivated chiefly by the examples

$$h \equiv 0,\ h_H = \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M H d\mu},\ h_{|H|} = \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M |H| d\mu}, \text{ and } h_K = \frac{-\int_M (\Delta H) K d\mu}{\int_M K d\mu},$$

where K is the Gauss curvature of M_t.

The first is simply surface diffusion flow (SD). Using Vol M_t to denote the volume enclosed by M_t in $\mathbb{R}^{n+1}$ we compute

$$\frac{d}{dt}\text{Vol } M_t = \int_M \Delta H d\mu = 0, \text{ and}$$
$$\frac{d}{dt}\int_M d\mu = \int_M H\Delta H d\mu = -\int_M \|\nabla H\|^2 d\mu \le 0;$$

so that a manifold evolving by (SD) will exhibit conservation of enclosed volume and monotonic decreasing surface area. Further, surface area is preserved exactly when the mean curvature of M_t is constant. It is these geometric characteristics of

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the surface diffusion flow which motivate the generalisation to constrained surface diffusion flows. For example with $h = h_H$, while $\int_M H d\mu \neq 0$ we have

$$\begin{aligned}\frac{d}{dt}\int_M d\mu &= \int_M H\Delta H d\mu + h_H \int_M H d\mu \\ &= -\int_M \|\nabla H\|^2 d\mu + \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M H d\mu}\int_M H d\mu = 0,\end{aligned}$$

and now surface area is conserved. Volume is monotonic increasing or decreasing depending on the sign of $\int_M H d\mu$, and preserved only when H is constant. Unfortunately, it seems more difficult to show that qualities such as convexity are preserved. This is due to the absence of a maximum principle. In particular, $\int_M H d\mu$ could approach zero under (CSD) with $h = h_H$, which would cause the flow to be undefined, and most likely *without* a curvature singularity. This motivates the use of $h_{|H|}$, where we replace the denominator with total mean curvature $\int_M |H| d\mu$. For this flow we compute

$$\begin{aligned}\frac{d}{dt}\text{Vol } M_t &= \int_M (\Delta H + h_{|H|}) d\mu = |M| \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M |H| d\mu} \geq 0, \quad \text{and} \\ \frac{d}{dt}\int_M d\mu &= \int_M H(\Delta H + h_{|H|}) d\mu \\ &= -\int_M \|\nabla H\|^2 d\mu + \int_M \|\nabla H\|^2 d\mu \frac{\int_M H d\mu}{\int_M |H| d\mu} \leq 0.\end{aligned}$$

Here enclosed volume and surface area are monotonic increasing and decreasing respectively. We also have not only that surface area is stationary (constant in time) if H is constant, but volume also. It can also be observed that if volume is constant, then surface area is necessarily constant. This is in contrast to (SD) flow, where volume is constant regardless of the behaviour of the surface area. Further, the flow speed itself is non-zero for surfaces of piecewise linear mean curvature. This leads us to believe that singularity development and asymptotic behaviour under (CSD) flow with $h = h_{|H|}$ will be easier to understand compared with that of (SD) flow. (Consider for example a clothoid-type manifold.) Finally, we use an inequality of Burago-Zalgaller [3] to infer

$$\int_M |H| d\mu \geq c_{BZ} |M_t|^{\frac{n}{n-1}} \geq c_{BZ}\big(\text{Vol } M_0\big) > 0,$$

where we also used the isoperimetric inequality and the fact that volume is monotonic increasing under this flow.

Following a similar line of reasoning gives rise to several other 'conservation' type flows. For example, with $h = h_K$ we calculate

$$\begin{aligned}\frac{d}{dt}\int_M H d\mu &= \int_M \big[(H^2 - \|A\|^2)(\Delta H + h_K) - \Delta^2 H\big] d\mu \\ &= \int_M K(\Delta H) d\mu + h_K \int_M K d\mu = 0,\end{aligned}$$

where $\|A\|^2$ denotes the squared norm of the second fundamental form of M_t. Thus the generalised mixed volume $\int_M H d\mu$ is always preserved under (CSD) flow with $h = h_K$. In this case $\int_M K d\mu$ is the denominator of h_K, which is constant under the flow, and so similarly to $h_{|H|}$ the constraint function h_K is always defined. One

expects that global analysis of flows such as this, which preserve a geometrically interesting quantity or keep it monotone in time, would lead to new geometric inequalities. At the very least we would expect to obtain new proofs of classical geometric inequalities, such as the isoperimetric inequality. This is in direct analogy with the work of Huisken [15] and the first author [26, 27, 28] for example.

A first step in any program of analysis for these flows is a short time existence theorem. The first appearance of such a theorem in the context of geometric heat flows in the literature is due to Huisken-Polden [16, 31]. While the idea of proof there is clear, the usage of the linearisation is not. This was later clarified in a much more restricted case by Sharples [32], who considers only second order flows, but claims the techniques are applicable also to the higher order case [33]. Independently, Escher, Mayer and Simonett [10] apply theorems credited to Amann to conclude short time existence for (SD) flow. Unfortunately the quoted references are not readily available.

Despite this confusion, there is a much more standard approach to the problem of short time existence for our flows (CSD) pointed out by Kuwert [18]. One may adapt the existence and uniqueness theory for higher order quasilinear parabolic partial differential equations in $\mathbb{R}^n$. This is easily accomplished by writing the problem as a graph, and then we must consider a degenerate quasilinear fourth order parabolic partial differential equation.

Depending on the constraint function, short time existence for this equation with f_0 at least $C^4(M_0)$ follows from (for example) the linear estimates found in Eidel'man and Zhitarashu [8], Solonnikov [36], or an extension of those in Friedman [12], combined with a fixed point argument. Uniqueness can be obtained by a method similar to that found in Li [22], which is originally due to unpublished notes of Amann. The relevant theorem is also stated in Amann [1].

Now, depending on the constraint function h there are two possible approaches: if we have a known function of t, such as $\frac{1}{1+t}$, $\sin t$, and so on, then one must show that $\partial_t h(0)$ is bounded. Otherwise, if we have a constraint function consisting of integrals as above with h_K and h_H, we use the initial smoothness of the immersion f_0 to guarantee estimates for h. For example, in the case where $h = h_H$, there are up to seven derivatives of the immersion in $\partial_t h$, and so if $f_0 \in C^7(M_0)$, we will have a short time existence theorem.

Therefore one can see that the regularity of f_0 required to obtain short time existence is at least $C^4(M_0)$, and if h consists of integrals of curvature then the required regularity could be quite high, depending on h. This is what we mean below when we say 'smooth enough'.

Theorem 1 (Short time existence)**.** *For any smooth enough initial immersion $f_0 : M^n \to \mathbb{R}^{n+1}$ and constraint function $h : I \to \mathbb{R}$ with I an interval containing* 0 *and $h \in C^1(I)$, there exists a unique nonextendable smooth solution $f : M \times [0, T) \to \mathbb{R}^{n+1}$ to* (CSD) *with $f(\cdot, 0) = f_0$, where $0 < T \leq \infty$.*

For $T \in (0, \infty]$ a solution $f : M \times [0, T) \to \mathbb{R}^{n+1}$ to (CSD) is nonextendable if $T \neq \infty$ and there is no $\delta > 0$ such that $\tilde{f} : M \times [0, T + \delta) \to \mathbb{R}^{n+1}$ is also a solution to (CSD) with $\tilde{f}(\cdot, 0) = f_0$. For the interested reader, a more detailed discussion of this theorem can be found in [39].

Motivated by the observation that (SD) flow can also be derived by considering the H^{-1}-gradient flow for the area functional (see Fife [11]), and the recent work

of Kuwert & Schätzle [19, 20] on the gradient flow for the Willmore functional, we present the following theorem.

Theorem 2 (Lifespan Theorem)**.** *Suppose* $n \in \{2,3\}$ *and let* $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ *be a closed immersion with* C^∞ *initial data evolving by*

$$\frac{\partial}{\partial t} f = (\Delta H + h)\nu. \tag{CSD}$$

Then there are constants $\rho > 0$, $\epsilon_0 > 0$, *and* $c < \infty$ *such that if* $h : I \supset [0, \frac{1}{c}\rho^4] \to \mathbb{R}$ *is a function satisfying*

$$\|h\|_{\infty,I} < \infty, \tag{A1}$$

and ρ *is chosen with*

$$\int_{f^{-1}(B_\rho(x))} \|A\|^n d\mu\Big|_{t=0} = \epsilon(x) \le \epsilon_0 \qquad \text{for any } x \in \mathbb{R}^{n+1}, \tag{1}$$

then for $n = 2$ *the maximal time* T *of smooth existence for the flow* (CSD) *with initial data* $f_0 = f(\cdot, 0)$ *satisfies*

$$T \ge \frac{1}{c}\rho^4, \tag{2}$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} \|A\|^n d\mu \le c\epsilon(x) \qquad \text{for} \qquad 0 \le t \le \frac{1}{c}\rho^4. \tag{3}$$

For $n = 3$, *the conclusions* (2), (3) *hold under the additional assumption that there exists an absolute constant* $C_{AB} \in (0, \infty)$ *such that*

$$|M_t| \le C_{AB}, \qquad \text{for} \qquad 0 \le t \le \frac{1}{c}\rho^4. \tag{AB}$$

The restriction on the dimension of the evolving immersion is due to both the exponent in the Michael-Simon Sobolev inequality, and the scaling of the total squared curvature functional. For flows where the evolution of the surface area is bounded (such as (SD) and (CSD) with $h = h_{|H|}$) we have removed the latter restriction by considering (1), which is a natural generalisation of (1.4) in [20]. The size of ϵ_0 is determined indirectly by the bound on surface area for the flow in question. As to the exponent in the Michael-Simon Sobolev inequality, the interplay between the evolution equations and our techniques using integral estimates forces $n < 4$; see Section 5 for a discussion of this issue.

At first glance, the choice in (1) may appear somewhat restrictive, since ϵ_0 (the size of which is dictated by estimates to come) may be very small. However, it is clear that if the initial surface M_0 is of finite total curvature (that is, $\int_M \|A\|^n d\mu\big|_{t=0} < \infty$), then there will exist a positive $\rho = \rho(\epsilon_0, M_0)$ such that (1) is satisfied. Therefore, in terms of allowable initial surfaces M_0, we are only excluding those for which the total curvature is infinite.

The assumption (A1) also appears restrictive. For an a priori known function of time, it is appropriate, but for our given examples ($h = h_{|H|}, h_K$) it is not clear that (A1) is satisfied. We will show in Section 3 that constraint functions similar to $h_{|H|}$ admit an a priori bound, while constraint functions of a form similar to h_K remain just beyond our current techniques. It is in this sense which the two examples

serve to differentiate between those constraint functions which are relatively easy to handle, and those which just present difficulty. The inequality

$$\sup_{x,y\in f(M)} |x-y| = \text{extrinsic diameter} = d_{ext} \le c_T(n)\int_M |H|^{n-1}d\mu$$

due to Topping [38] will also play a major role, allowing us to prescribe a class of constraint functions which admit a 'localisation' procedure. The extra assumptions required will be a growth condition, and a geometric condition: either bounded surface area or bounded total mean curvature.

In a more global sense, we present the lifespan theorem with a perspective toward further analysis of the (CSD) flows. In particular, as the statement depends on the concentration of the curvature of the initial surface, the result is particularly relevant to the analysis of asymptotic behaviour in the following respect. When considering a blowup of a singularity formed at some time $T < \infty$ of the (CSD) flow, we wish to have that some amount of the curvature concentrates in space. From the theorem, if $\rho(t)$ denotes the largest radius such that (1) holds at time t, then $\rho(t) \le \sqrt[4]{c(T-t)}$ and so at least ϵ_0 of the curvature concentrates in a ball $f^{-1}(B_{\rho(T)}(x))$. That is,

$$\lim_{t\to T}\int_{f^{-1}(B_{\rho(t)}(x))} \|A\|^n d\mu \ge \epsilon_0,$$

where $x = x(t)$ is understood to be the centre of a ball where the integral above is maximised.

As already mentioned, our motivation for the extension of (SD) to the more general class of flows (CSD) is essentially mathematical. However there does already exist a large body of work on (SD) flow itself, and study of (SD) alone is well motivated. First proposed by the physicist Mullins [30] in 1957 (two years before he proposed the mean curvature flow), it was originally designed to model the formation of tiny thermal grooves in phase interfaces where the contribution due to evaporation-condensation was insignificant. Some time later, Davi, Gurtin, Cahn and Taylor [5, 7] proposed many other physical models which give rise to the surface diffusion flow. These all exhibit a reduction of free surface energy and conservation of volume; an essential characteristic of (SD) flow. There are also other motivations for the study of (SD). For example, two years later Cahn, Elliot and Novick-Cohen [4] proved that (SD) is the singular limit of the Cahn-Hilliard equation with a concentration dependent mobility. Among other applications, this arises in the modeling of isothermal separation of compound materials.

Analysis of the surface diffusion flow began slowly, with the first works appearing in the early 80s. Baras, Duchon and Robert [2] showed the global existence of weak solutions for two dimensional strip-like domains in 1984. Later, in 1997 Elliot and Garcke [9] analysed (SD) flow of curves, and obtained local existence and regularity for C^4-initial curves, and global existence for small perturbations of circles. Significantly, Ito [17] showed in 1998 that convexity will not be preserved under (SD), even for smooth, rotationally symmetric, closed, compact, strictly convex initial hypersurfaces. In contrast with the case for second order flows such as mean curvature flow, this behaviour appears pathological. Escher, Mayer and Simonett [10] gave several numerical schemes for modeling (SD) flow, and have also given the only two known numerical examples [25] of the development of a singularity: a tubular spiral and thin-necked dumbbell. They also provide an example of an immersion

which will self-intersect under the flow, a figure eight knot. In 2001, Simonett [35] used centre manifold techniques to show that for initial data $C^{2,\alpha}$-close to a sphere, both the surface diffusion and Willmore flows (Willmore flow in one codimension is $\partial_t f = \Delta H + \|A^o\|^2 H$, where $A^o = A - \text{trace}_g\ A$) exist for all time and converge asymptotically to a sphere.

There have been many important works on fourth order flows of a slightly different character, from Willmore flow of surfaces to Calabi flow, a fourth order flow of metrics. Significant contributions to the analysis of these flows by the authors Kuwert, Schätzle, Polden, Huisken, Mantegazza and Chruściel [6, 19, 20, 24, 31] are particularly relevant, as the methods employed there are similar to ours here.

In our proof, we exploit the fact that for an n-dimensional immersion the integral

$$\int_M \|A\|^n d\mu$$

is scale invariant. The technique used by Struwe [37] is then relevant, although as with all higher order flows the major difficulty is in overcoming the lack of powerful techniques unique to the second order case. In particular, we are without the maximum principle, and this implies that the geometry of the surface could deteriorate, as in [17]. Therefore we are forced to use integral estimates to derive derivative curvature bounds under a condition similar to (1), and in calculating these estimates it is crucial to only use inequalities which involve universal constants. Interpolation inequalities similar in nature to those used by Ladyzhenskaya, Ural'tseva and Solonnikov [21] and Hamilton [13], and the Sobolev inequality of Michael-Simon [29], are invaluable in this regard.

The structure of this paper is as follows. To apply the argument used by Struwe, we must prove two key local integral estimates. In Section 2 we collect various fundamental formulae from differential geometry, set our notation, and state some basic results. The goal of Section 3 is to show that the a priori bound (A1) is satisfied by a class of constraint functions, and to detail the localisation procedure required to use the global constraint function in local integral estimates. Section 4 is concerned with estimating the evolution of local integrals of derivatives of curvature. Section 5 combines these estimates with Sobolev inequalities, interpolation inequalities, and the results of Section 3 to conclude the two required key integral estimates. With these in hand, we adapt the argument of Struwe in Section 6 to prove the lifespan theorem. Section 7 contains some remarks on lifespan theorems for flows similar to (CSD).

We note that a theorem similar to Theorem 2 was proposed in [23], applying only to the flow (SD). Our work includes a proof of this result, when $n = 2$ and $n = 3$.

2. Notation and preliminary results.

In this section we will collect various general formulae from differential geometry which we will need when performing the later analysis. We will adopt similar notation to Hamilton [13] and Huisken [14]. We have as our principal object of study a smooth immersion $f : M^n \to \mathbb{R}^{n+1}$ of an orientable compact hypersurface M, and induced metric tensor with components

$$g_{ij} = \left(\frac{\partial}{\partial x_i} f \,\middle|\, \frac{\partial}{\partial x_j} f \right),$$

so that the pair (M, g) is a Riemannian manifold. In the above equation $(\cdot|\cdot)$ denotes the regular Euclidean inner product, and $\frac{\partial}{\partial x_i}$ is the derivative in the direction of the i-th basis vector of the ambient space, which in our case is the regular Euclidean partial derivative. When convenient we frequently use the abbreviation $\partial_i = \frac{\partial}{\partial x_i}$.

The Riemannian metric induces an inner product structure on all tensors, which we define as the trace over pairs of indices with the metric:

$$\left\langle T^i_{jk}, S^i_{jk} \right\rangle = g_{is} g^{jr} g^{ku} T^i_{jk} S^s_{ru}, \qquad \|T\|^2 = \langle T, T\rangle\,,$$

where repeated indices are summed over from 1 to n. The mean curvature H is defined by

$$H = g^{ij} A_{ij} = A^i_i,$$

where the components A_{ij} of the second fundamental form A are given by

$$A_{ij} = -\left(\frac{\partial^2}{\partial x_i \partial x_j} f \,\middle|\, \nu \right) = \left(\frac{\partial}{\partial x_j} f \,\middle|\, \frac{\partial}{\partial x_i} \nu \right), \tag{4}$$

where ν is the outer unit normal vector field on M.

The Christoffel symbols of the induced connection are determined by the metric,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij} \right), \tag{5}$$

so that then the covariant derivative on M of a vector X and of a covector Y is

$$\begin{aligned} \nabla_j X^i &= \frac{\partial}{\partial x_j} X^i + \Gamma^i_{jk} X^k, \text{ and} \\ \nabla_j Y_i &= \frac{\partial}{\partial x_j} Y_i - \Gamma^k_{ij} Y_k \end{aligned}$$

respectively.

From the expression (4) and the smoothness of f we can see that the second fundamental form is symmetric; less obvious but equally important is the symmetry of the first covariant derivatives of A,

$$\nabla_i A_{jk} = \nabla_j A_{ik} = \nabla_k A_{ij},$$

commonly referred to as the Codazzi equations.

The fundamental relations between components of the Riemann curvature tensor R_{ijkl}, the Ricci tensor R_{ij} and scalar curvature R are given by Gauss' equation

$$R_{ijkl} = A_{ik} A_{jl} - A_{il} A_{jk},$$

with contractions

$$\begin{aligned} g^{jl} R_{ijkl} &= R_{ik} = H A_{ik} - A^j_i A^k_j, \text{ and} \\ g^{ik} R_{ik} &= R = H^2 - \|A\|^2. \end{aligned}$$

We will need to interchange covariant derivatives; for vectors X and covectors Y we obtain

$$\begin{aligned} \nabla_{ij} X^h - \nabla_{ji} X^h &= R^h_{ijk} X^k = (A_{lj} A_{ik} - A_{lk} A_{ij}) g^{hl} X^k, \\ \nabla_{ij} Y_k - \nabla_{ji} Y_k &= R_{ijkl} g^{lm} Y_m = (A_{lj} A_{ik} - A_{il} A_{jk}) g^{lm} Y_m, \end{aligned}$$

where $\nabla_{i_1 \ldots i_n} = \nabla_{i_1} \cdots \nabla_{i_n}$. Further we define $\nabla_{(n)} T$ to be the tensor with components $\nabla_{i_1 \ldots i_n} T^{k_1 \ldots}_{j_1 \ldots}$. We also use for tensors T and S the notation $T * S$ (as in

Hamilton [13]) to denote a linear combination of new tensors, each formed by contracting pairs of indices from T and S by the metric g with multiplication by a universal constant. The resultant tensor will have the same type as the other quantities in the equation it appears. Keeping these in mind we also denote polynomials in the iterated covariant derivatives of these terms by

$$P_j^i(T) = \sum_{k_1+\ldots+k_j=i} c\nabla_{(k_1)}T * \cdots * \nabla_{(k_j)}T,$$

where the constant $c \in \mathbb{R}$ is absolute and may vary from one term in the summation to another. As is common for the $*$-notation, we slightly abuse this constant when certain subterms do not appear in our P-style terms. For example

$$\begin{aligned}\|\nabla A\|^2 &= \langle \nabla A, \nabla A\rangle \\ &= 1 \cdot \big(\nabla_{(1)}A * \nabla_{(1)}A\big) + 0 \cdot \big(A * \nabla_{(2)}A\big) \\ &= P_2^2(A).\end{aligned}$$

This will occur throughout the paper without further comment.

The Laplacian we will use is the Laplace-Beltrami operator on M, with the components of ΔT given by

$$\Delta T^i_{jk} = g^{pq}\nabla_{pq}T^i_{jk} = \nabla^p\nabla_p T^i_{jk}.$$

Using the Codazzi equation with the interchange of covariant derivative formula given above, we obtain Simons' identity:

$$\begin{aligned}\Delta A_{ij} &= \nabla_{ij}H + HA_{il}g^{lm}A_{mj} - \|A\|^2 A_{ij} \\ &= \nabla_{ij}H + HA_i^l A_{lj} - \|A\|^2 A_{ij},\end{aligned}$$

or in $*$-notation

$$\Delta A = \nabla_{(2)}H + A * A * A. \tag{SI}$$

In the coming sections we will be concerned with calculating the evolution of the iterated covariant derivatives of curvature quantities. The following less precise interchange of covariant derivatives formula (derived from the fundamental equations above) will be useful to keep in mind:

$$\nabla_{ij}T = \nabla_{ji}T + P_2^0(A) * T.$$

In most of our integral estimates (especially those in sections 4 and 5), we will be including a function $\gamma : M \to \mathbb{R}$ in the integrand. Eventually, this will be specialised to a smooth cutoff function between concentric geodesic balls on M. For these estimates however, we will only assume that $\gamma = \tilde{\gamma} \circ f$, where

$$0 \le \tilde{\gamma} \le 1, \qquad \text{and} \qquad \|\tilde{\gamma}\|_{C^2(\mathbb{R}^{n+1})} \le c_{\tilde{\gamma}} < \infty.$$

Using the chain rule, this implies $D\gamma = (D\tilde{\gamma} \circ f)Df$ and then $D^2\gamma = (D^2\tilde{\gamma} \circ f)(Df, Df) + (D\tilde{\gamma} \circ f)D^2 f(\cdot,\cdot)$. Using the expression (5) for the Christoffel symbols to convert the computations above to covariant derivatives, and the Weingarten relations

$$\partial_i \nu = A_i^j \partial_j f, \qquad \partial_i\partial_j f = -A_{ij}\nu,$$

to convert the derivatives of ν to factors of the second fundamental form with the basis vectors $\partial_i f$, we obtain the estimates

$$\|\nabla\gamma\| \le c_{\gamma 1}, \qquad \text{and} \qquad \|\nabla_{(2)}\gamma\| \le c_{\gamma 2}(1 + \|A\|). \tag{γ}$$

For a given $\rho > 0$, we also define the functions $\epsilon, \delta^{(p)} : \mathbb{R}^{n+1} \times [0, T^*] \to \mathbb{R}$ as

$$\epsilon(x) = \int_{f^{-1}(B_\rho(x))} \|A\|^2 d\mu, \quad \text{and} \quad \delta^{(p)}(x) = \int_{f^{-1}(B_\rho(x))} \|A\|^p d\mu.$$

At times we will instead consider the set $[\gamma > 0] = \{q \in M : \gamma(q) > 0\}$ as the domain of the integrals in $\epsilon(x)$ and $\delta^{(p)}(x)$.

3. A priori estimates for the constraint function.

Our constraint functions are by their nature global notions (being functions of time only). This is a distinct advantage in some areas of the analysis: evolution equations first order in time and of any order in space involve at most a linear factor of h.

When one wishes to prove local integral estimates however, the global nature of h becomes an issue. We are faced with situations such as

$$\begin{aligned} &\frac{d}{dt} \int_{f^{-1}(B_\rho(x))} \|A\|^2 d\mu + \int_{f^{-1}(B_\rho(x))} \|\nabla_{(2)} A\|^2 d\mu \\ &\qquad \le h \int_{f^{-1}(B_{2\rho}(x))} \big(\|A\|^3 + \|A\|^2\big) d\mu + \text{“good terms”}, \end{aligned} \tag{6}$$

armed with a local smallness of curvature assumption

$$\sup_{\substack{x \in \mathbb{R}^{n+1} \\ t \in [0,T^*]}} \epsilon(x) \le \epsilon_0, \quad \text{or} \quad \sup_{\substack{x \in \mathbb{R}^{n+1} \\ t \in [0,T^*]}} \delta^{(p)}(x) \le \delta_0,$$

and tasked with absorbing the term involving h, a global term, into

$$\int_{f^{-1}(B_\rho(x))} \|\nabla_{(2)} A\|^2 d\mu,$$

a local integral. Assume for the sake of example that $h = \int_M k(\mathcal{W}) d\mu$, where $\mathcal{W}$ is the Weingarten map, and h obeys an estimate

$$h \le C_{ABS} \int_M \|A\|^2 d\mu \int_M \|\nabla_{(2)} A\|^2 d\mu,$$

where C_{ABS} is an absolute constant. Then as a first attempt to ‘localise’ the integrals on the right one might estimate them by

$$\begin{aligned} &\int_M \|A\|^2 d\mu \int_M \|\nabla_{(2)} A\|^2 d\mu \\ &\qquad \le c_\rho^2(t) \sup_{x \in \mathbb{R}^{n+1}} \int_{f^{-1}(B_\rho(x))} \|A\|^2 d\mu \sup_{x \in \mathbb{R}^{n+1}} \int_{f^{-1}(B_\rho(x))} \|\nabla_{(2)} A\|^2 d\mu \\ &\qquad \le c_\rho^2(t) \epsilon_0 \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu, \end{aligned}$$

where $c_\rho(t)$ is the number of extrinsic balls of radius ρ required to cover $f(M_t)$ and $x_1 \in \mathbb{R}^{n+1}$ is a point where the second supremum is attained. The goal of course is to now bound $c_\rho^2(t)\epsilon_0$ by $\frac{1}{2C_{ABS}}$ (for example), and absorb the entire term on the left in (6). Unfortunately, this will in general not be possible. To attain a smaller ϵ_0, one must drive ρ to zero, but this will in turn drive c_ρ to ∞. Further, the scaling is unfavourable, making it difficult to know a priori if any admissible $\rho > 0$ exists. Finally, c_ρ is a function of time, and without a uniform bound we have little hope of absorbing the constraint function into a local integral.

With some minor modifications to the above idea, and assumptions on the flow, these problems can be overcome and the argument carries through. Our main result for this section is the following.

Theorem 3. *Let $f : M^n \times [0, T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with constraint function h satisfying for some $j, k, l \in \mathbb{N}_0$*

$$h \le \int_M P_j^2(A) + P_k^1(A) + P_l^0(A)d\mu \tag{A2}$$

where for $m = \max\{2k-2, 2j-k, l, n^2+n-2\}$

$$\sup_{x\in\mathbb{R}^{n+1}} \delta^{(m)}(x) \le \delta_0^{(m)} < \infty,$$

and for a finite absolute constant C_{AB}

$$|M_t| \le C_{AB}; \tag{AB}$$

on $[0, T^*]$.

Then for any $\rho > 0$, $x \in \mathbb{R}^{n+1}$, $t \in [0, T^*]$ there exists an $x_1 \in \mathbb{R}^{n+1}$ such that for any $\theta > 0$,

$$h\int_{f^{-1}(B_{2\rho}(x))} \big(\|A\|^4 + \|A\|^2\big)d\mu \le \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + C_{UGLY},$$

if $j, k \neq 0$, and otherwise

$$h\int_{f^{-1}(B_{2\rho}(x))} \big(\|A\|^4 + \|A\|^2\big)d\mu \le C_{UGLY},$$

where $C_{UGLY} = C_{UGLY}(\theta, \delta_0^{(m)}, C_{AB}, \rho, j, k, l, n)$.

Before we begin the proof we would like to show that $h_{|H|}$ satisfies the assumptions of the theorem. By viewing mean curvature as the variation of area, Burago-Zalgaller [3] prove the estimate

$$|M| \le c\bigg(\int_M |H|d\mu\bigg)^{\frac{n}{n-1}} \tag{7}$$

for a constant c depending only on n. Using now the isoperimetric inequality we conclude

$$\frac{1}{\int_M |H|d\mu} \le c|M|^{\frac{1-n}{n}} \le c(\text{Vol } M)^{-1} \le c\text{Vol } M_0.$$

Therefore we may estimate

$$h_{|H|}(t) = \frac{\int_M \|\nabla H\|^2 d\mu}{\int_M |H|d\mu} \le c(M_0)\int_M P_2^2(A)d\mu.$$

Thus for any dimension n we take $m = (n-1)(n+2)$. Also, (AB) is satisfied with

$$C_{AB} = |M_0|.$$

Driving Theorem 3 is the following estimate due to Topping [38].

Theorem 4. *Let M^n be a compact connected n-dimensional submanifold of $\mathbb{R}^{n+1}$. Then its extrinsic diameter and its mean curvature H are related by*

$$d_{ext} \le c_T(n)\int_M |H|^{n-1}d\mu.$$

Topping shows that in particular we may take $c_T(2) = \frac{32}{\pi}$. We refer the reader to the references in [38] and [34] for a history of this inequality and others similar to it.

We first obtain an estimate for $c_\rho(t)$.

Lemma 5. *Let $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow satisfying* (AB)*. Then for any ρ such that $0 < \rho \le \frac{d_{ext}\sqrt{n+1}}{2}$ there exists an $x_2 \in \mathbb{R}^{n+1}$ where the following estimate holds:*

$$c_\rho(t) \le c(C_{AB}, \rho, n)\bigg(\int_{f^{-1}(B_\rho(x_2))} \|A\|^{(n-1)(n+2)} d\mu\bigg)^{n+1}.$$

Remark. If $\rho > \frac{d_{ext}\sqrt{n+1}}{2}$ then $c_\rho(t) = 1$. We will always assume from now on that $0 < \rho \le \frac{d_{ext}\sqrt{n+1}}{2}$.

Proof. We simply apply a covering argument, Theorem 4, and then the Hölder inequality. Since we can cover M_t by an $(n+1)$-cube with side length d_{ext} and a ball of radius ρ encloses an $(n+1)$-cube with side length $\frac{2\rho}{\sqrt{n+1}}$,

$$\begin{aligned}
c_\rho(t) &\le \left(\frac{d_{ext}\sqrt{n+1}}{2\rho}\right)^{n+1} \\
&\le \left(\frac{c_T(n)\sqrt{n+1}}{2\rho}\right)^{n+1}\bigg(\int_M |H|^{n-1} d\mu\bigg)^{n+1} \\
&\le \left(\frac{c_T(n)\sqrt{n+1}}{2\rho}\right)^{n+1} |M_t|^{\frac{(n+1)^2}{n+2}}\bigg(\int_M |H|^{(n-1)(n+2)} d\mu\bigg)^{\frac{n+1}{n+2}} \\
&\le \left(\frac{c_T(n)\sqrt{n+1}}{2\rho}\right)^{n+1} |M_t|^{\frac{(n+1)^2}{n+2}}\bigg(\sup_{x\in\mathbb{R}^{n+1}} c_\rho(t)\int_{f^{-1}(B_\rho(x))} |H|^{(n-1)(n+2)} d\mu\bigg)^{\frac{n+1}{n+2}},
\end{aligned}$$

so

$$c_\rho(t) \le \left(\frac{c_T(n)\sqrt{n+1}}{2\rho}\right)^{(n+1)(n+2)} C_{AB}^{(n+1)^2}\bigg(\int_{f^{-1}(B_\rho(x_2))} \|A\|^{(n-1)(n+2)} d\mu\bigg)^{n+1},$$

where x_2 is a point in $\mathbb{R}^{n+1}$ such that

$$\int_{f^{-1}(B_\rho(x_2))} \|A\|^{(n-1)(n+2)} d\mu = \sup_{x\in\mathbb{R}^{n+1}} \int_{f^{-1}(B_\rho(x))} \|A\|^{(n-1)(n+2)} d\mu.$$

□

Remark. Since we can take $c_T(2) = \frac{32}{\pi}$, the conclusion in the theorem above for a (CSD) flow with $h = h_{|H|}$ and $n = 2$ is

$$c_\rho(t) \le \left(\frac{32\sqrt{3}}{2\pi\rho}\right)^{12} |M_0|^9 \bigg(\int_{f^{-1}(B_\rho(x_2))} \|A\|^4 d\mu\bigg)^3.$$

We now use the above to estimate h.

Lemma 6. *Let $\theta > 0$ be a fixed positive number and $f : M \times [0,T^*] \to \mathbb{R}^{n+1}$ a* (CSD) *flow satisfying the assumptions of Theorem 3. Then for any $\rho > 0$ there*

exists a point $x_1 \in \mathbb{R}^{n+1}$ such that the constraint function h satisfies the following estimate:

$$h \le \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c(\theta,\rho,n,j,k,l,C_{AB},\delta_0^{(m)})\delta_0^{(m)}$$

for $j,k,l \ne 0$, and

$$h \le c(\rho,n,C_{AB})\big(\delta_0^{(m)}\big)^{n+1}$$

for $j=k=l=0$.

Proof. Recall that

$$\sup_{x\in\mathbb{R}^{n+1}} \delta^{(m)}(x) \le \delta_0^{(m)} < \infty,$$

where $m = \max\{2j-2, 2k-j, l, n^2+n-2, 4\}$.

We will first prove the estimate assuming that $j \ge \max\{2, 2k+1\}$:

$$\begin{aligned}
h &\le \int_M P_j^2(A) + P_k^1(A) + P_l^0(A) d\mu \\
&\le c \sup_{x\in\mathbb{R}^{n+1}} \Big(c_\rho \int_{f^{-1}(B_\rho(x))} \|\nabla_{(2)} A\| \cdot \|A\|^{j-1} d\mu \Big) + \int_M \|\nabla A\|^2 \|A\|^{j-2} d\mu \\
&\quad + c \int_M \|\nabla A\| \cdot \|A\|^{k-1} d\mu + c\int_M \|A\|^l d\mu \\
&\le \frac{\theta}{2} \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c_\rho^2 \frac{c}{2\theta} \int_{f^{-1}(B_\rho(x_1))} \|A\|^{2j-2} d\mu \\
&\quad + c\int_M \|\nabla A\|^2 \|A\|^{j-2} d\mu + c\int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\le \frac{\theta}{2} \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c\Big(1 + (j-2)\frac{4-j}{j-3}\Big) \int_M \langle A, \Delta A\rangle \|A\|^{j-2} d\mu \\
&\quad + c_\rho^2 \frac{c}{2\theta} \int_{f^{-1}(B_\rho(x_1))} \|A\|^{2j-2} d\mu + c(\theta,j,k,l) \int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\le \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c_\rho^2 c(\theta,j,k) \int_{f^{-1}(B_\rho(x_1))} \|A\|^{2j-2} d\mu \\
&\quad + c(\theta,j,k,l) \int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\le \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c_\rho^2 c(\theta,j,k,C_{AB}) \Big(\int_{f^{-1}(B_\rho(x_1))} \|A\|^m d\mu \Big)^{\frac{2j-2}{m}} \\
&\quad + c(\theta,j,k,l,C_{AB}) \Big(\sup_{x\in\mathbb{R}^{n+1}} c_\rho \int_{f^{-1}(B_\rho(x))} \|A\|^m d\mu \Big)^{\frac{2k-j+l}{m}} \\
&\le \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c_\rho^{\frac{2m+2k-j+l}{m}} c(\theta,j,k,l,C_{AB}) \big(\delta_0^{(m)}\big)^{\frac{j-2+2k+l}{m}} \\
&\le \theta \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c(\theta,\rho,n,j,k,l,C_{AB}) \big(\delta_0^{(m)}\big)^{\frac{(n+1)(2m+2k-j+l)+j-2+2k+l}{m}}.
\end{aligned}$$

The estimate is easier to prove in the subcases excluded above. When $j = 1$ we instead split the first integral by

$$\begin{aligned}\int_M P_j^2(A)d\mu &\le c \sup_{x\in\mathbb{R}^{n+1}} c_\rho \int_{f^{-1}(B_\rho(x))} \|\nabla_{(2)}A\| d\mu \\ &\le \frac{\theta}{2}\int_{f^{-1}(B_\rho(x_3))} \|\nabla_{(2)}A\|^2 d\mu + c_\rho^2\frac{2}{\theta}\int_{f^{-1}(B_\rho(x_3))} 1\, d\mu \\ &\le \frac{\theta}{2}\int_{f^{-1}(B_\rho(x_3))} \|\nabla_{(2)}A\|^2 d\mu + c(\theta,\rho,n,C_{AB})\big(\delta_0^{(m)}\big)^{2n+2}.\end{aligned}$$

When $j < 2k+1$ we instead estimate the second integral by

$$\begin{aligned}\int_M P_k^2(A)d\mu &\le c\int_M \|\nabla A\|\cdot\|A\|^{k-1} d\mu \\ &\le c\int_M \|\nabla A\|^2 d\mu + c\int_M \|A\|^{2k-2} d\mu \\ &\le c\int_M \|\nabla_{(2)}A\|\cdot\|A\| d\mu + c\int_M \|A\|^{2k-2} d\mu \\ &\le \frac{\theta}{2}\int_{f^{-1}(B_\rho(x_3))} \|\nabla_{(2)}A\|^2 d\mu + c_\rho^2\frac{2}{\theta}\int_{f^{-1}(B_\rho(x_3))} \|A\|^2 d\mu + c\int_M \|A\|^{2k-2} d\mu \\ &\le \frac{\theta}{2}\int_{f^{-1}(B_\rho(x_3))} \|\nabla_{(2)}A\|^2 d\mu + c(\theta,\rho,n,C_{AB})\Big[\big(\delta_0^{(m)}\big)^{2n+2+\frac{2}{m}} + \big(\delta_0^{(m)}\big)^{\frac{2k-2}{m}}\Big].\end{aligned}$$

Note that in any case, the exponent of $\delta_0^{(m)}$ is greater than 1 due to the conditions on m. This gives the first part of the lemma.

If $j = k = l = 0$ then obviously

$$h \le c(\rho, n, C_{AB})\big(\delta_0^{(m)}\big)^{n+1}.$$

This finishes the proof. □

Remark. In the special case where $h = h_{|H|}$ and $n = 2$, the estimate reads

$$h_{|H|} \le \theta\int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)}A\|^2 d\mu + \frac{c_{BZ}}{4\theta\sqrt{\mathrm{Vol}\ M_0}}\left(\frac{16\sqrt{3}}{\rho\pi}\right)^2 4|M_0|^{\frac{37}{2}}\big(\delta_0^{(4)}\big)^{\frac{13}{2}},$$

where c_{BZ} is the constant from the inequality (7).

We are now ready to prove Theorem 3 as essentially a corollary to Lemma 6 above.

Proof of Theorem 3. First note that

$$\begin{aligned}\int_{f^{-1}(B_{2\rho}(x))} \big(\|A\|^4 + \|A\|^2\big)d\mu &\le \sup_{x^*\in B_{2\rho}(x)} 4^{n+1}\int_{f^{-1}(B_\rho(x^*))} \big(\|A\|^4 + \|A\|^2\big)d\mu \\ &\le 4^{n+1}C_{AB}^{1-\frac{4}{m}}\big(\delta_0^{(m)}\big)^{\frac{4}{m}}.\end{aligned}$$

By Lemma 6 we are now finished, choosing

$$\theta = \frac{\theta^*}{4^{n+1}C_{AB}^{1-\frac{4}{m}}\big(\delta_0^{(m)}\big)^{\frac{4}{m}}}.$$

□

Remark. In each of the previous inequalities we have been primarily concerned with integrals localised to a ball $f^{-1}(B_\rho(x))$. In the following sections where we derive the basic integral estimates, the domain of integration will instead be the set $[\gamma > 0]$, for γ as in (γ). This is necessary to not only obtain the local integral estimates, but also to allow us enough freedom to choose various appropriate γ functions, depending upon the situation. To bridge the gap between the two domains of integration we may choose $\gamma = \tilde{\gamma} \circ f$ to be such that

$$\chi_{B_\rho(x)} \le \tilde{\gamma} \le \chi_{B_{2\rho}(x)}$$

and $\gamma \in C^2(M)$. Then for a non-negative integrand we crudely estimate

$$\int_{f^{-1}(B_\rho(x))} [\cdots] d\mu \le \int_{[\gamma>0]} [\cdots] d\mu \le \int_{f^{-1}(B_{2\rho}(x))} [\cdots] d\mu.$$

This is why in Theorem 3 we see integrals with balls of radii 2ρ on the left.

Theorem 4 gives us the opportunity to obtain the derivative of curvature estimates in the ball $B_\rho(x_1)$, but nowhere else. This is not enough to prove the lifespan theorem. However, we may still proceed by using the estimates in the ball $B_\rho(x_1)$ to bound the constraint function over all of M_t, and then once this is accomplished we can go back and prove the required derivative of curvature estimates everywhere else on M_t.

Corollary 7 (The curvature estimates on a special ball)**.** *Suppose $n \in \{2, 3\}$ and let $f : M^n \times [0, T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with h satisfying the assumptions of Theorem 3. Then there is a $\delta_0^{(m)} = \delta_0^{(m)}(n, M_0)$ such that if*

$$\sup_{t\in[0,T^*],x\in\mathbb{R}^{n+1}} \int_{f^{-1}(B_\rho(x))} \|A\|^m d\mu \le \delta_0^{(m)},$$

there is an $x_1 \in \mathbb{R}^{n+1}$ such that

$$\|\nabla_{(2)}A\|^2_{\infty,f^{-1}(B_\rho(x_1))} \le c\big(\delta_0^{(m)}, T^*, C_{AB}, \rho, j, k, l, m, \alpha_0(2)\big),$$

where $\alpha_0(p) = \displaystyle\sum_{j=0}^{p} \sup_{x\in\mathbb{R}^{n+1}} \|\nabla_{(j)}A\|_{2,f^{-1}(B_\rho(x))}\Big|_{t=0}$.

Proof. Observe that the smallness assumption and (AB) implies that

$$\int_{f^{-1}(B_\rho(x))} \|A\|^n d\mu \le C_{AB}^{\frac{m-n}{m}} \left(\int_{f^{-1}(B_\rho(x)} \|A\|^m d\mu\right)^{\frac{n}{m}} \le C_{AB}^{\frac{m-n}{m}} \big(\delta_0^{(m)}\big)^{\frac{n}{m}} < \epsilon_0,$$

for

$$\delta_0^{(m)} < (\epsilon_0)^{\frac{m}{n}} C_{AB}^{\frac{n-m}{m}}.$$

Let γ be a cutoff function on M between a ball of radius ρ and a ball of radius 2ρ, as in the remark above. Then the smallness assumption (14) of Proposition 19 is satisfied for $\delta_0^{(m)}$ as above, that is

$$\sup_{[0,T^*]} \int_{f^{-1}(B_\rho(x))} \|A\|^n d\mu \le \epsilon_0.$$

Proposition 14 with $k = 0$ and our choice of γ gives:

$$\begin{aligned}\frac{d}{dt}\int_{f^{-1}(B_\rho(x))}\|A\|^2 d\mu + (2-\theta)\int_{f^{-1}(B_\rho(x))}\|\nabla_{(2)}A\|^2 d\mu \\ \le ch\int_{f^{-1}(B_{2\rho}(x))}([A*A]*A)\,d\mu + ch\int_{f^{-1}(B_{2\rho}(x))}\|A\|^2 d\mu \\ + c\int_{f^{-1}(B_{2\rho}(x))}\|A\|^2 d\mu + c\int_{f^{-1}(B_{2\rho}(x))}\big([P_3^2(A)+P_5^0(A)]*A\big)\,d\mu.\end{aligned}$$

Using Theorem 3 we obtain

$$\begin{aligned}\frac{d}{dt}\int_{f^{-1}(B_\rho(x_1))}\|A\|^2 d\mu + (2-\theta)\int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2 d\mu \\ \le c\delta_0^{(m)} + c\int_{f^{-1}(B_{2\rho}(x_1))}\big([P_3^2(A)+P_5^0(A)]*A\big)\,d\mu.\end{aligned}$$

Proceeding now exactly as in Proposition 19, we recover (15) for balls centred at the point x_1. Note that no constant depends on $\|h\|_\infty$. Moving on, we use the equation above to conclude (18) in the case where there are no derivatives of curvature, with no additional factors of the constraint function on the right hand side. That is,

$$\begin{aligned}\frac{d}{dt}\int_{f^{-1}(B_\rho(x_1))}\|A\|^2 d\mu + \frac{1}{2}\int_{f^{-1}(B_\rho(x_1))}\|\nabla_{(2)}A\|^2 d\mu \\ \le c\|A\|^2_{2,f^{-1}(B_{2\rho}(x_1))}(1+\|A\|^4_{\infty,f^{-1}(B_{2\rho}(x_1))}).\end{aligned}$$

Using this in the proof of Proposition 22 in place of Proposition 21 gives the required derivative of curvature bounds. □

Remark. Allowable choices of x_1 depend upon the splitting of integrals in Lemma 6, and this depends upon j, k and l. The proof of the next result will depend upon which class of allowable points is associated with the given constraint function.

We note that the assumption required is global, disguised as a local assumption. This is different to the case where we have no constraint function (such as for the surface diffusion or Willmore flows). However, even there, in the final argument used to prove the lifespan theorem one still requires this 'global disguised as local' assumption. We are merely introducing this concept earlier in the analysis.

Corollary 8 (The uniform bound for h)**.** *Suppose $n \in \{2,3\}$ and let $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with h satisfying the assumptions of Theorem 3. Then there is a $\delta_0^{(m)} = \delta_0^{(m)}(n, M_0)$ such that if*

$$\sup_{[0,T^*],x\in\mathbb{R}^{n+1}}\int_{f^{-1}(B_\rho(x))}\|A\|^m d\mu \le \delta_0^{(m)}, \tag{8}$$

the constraint function satisfies the estimate

$$\|h\|_{[0,T^*],\infty} \le c_h < \infty,$$

where $c_h = c_h(\delta_0^{(m)}, C_{AB}, \rho, j, k, l, n)$.

Proof. Using Corollary 7 above, we can directly estimate h by localising as in the proof of Lemma 6. This is however contingent upon us retrieving integrals around an allowable point $x_1 \in \mathbb{R}^{n+1}$ from the conclusion of Corollary 7. So we must be somewhat careful with our estimates below.

Firstly, for the case where $j \geq \max\{2, 2k+1\}$,

$$
\begin{aligned}
h &\leq \int_M P_j^2(A) + P_k^1(A) + P_l^0(A) d\mu \\
&\leq c_\rho c \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\| \cdot \|A\|^{j-1} d\mu + \int_M \|\nabla A\|^2 \|A\|^{j-2} d\mu \\
&\quad + c \int_M \|\nabla A\| \cdot \|A\|^{k-1} d\mu + c \int_M \|A\|^l d\mu \\
&\leq \frac{1}{2} \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c\Big(1 + (j-2)\frac{4-j}{j-3}\Big) \int_M \langle A, \Delta A \rangle \|A\|^{j-2} d\mu \\
&\quad + c_\rho^2 \frac{c}{2} \int_{f^{-1}(B_\rho(x_1))} \|A\|^{2j-2} d\mu + c(j,k,l) \int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\leq \frac{1}{2} \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c(j) \int_M \|\nabla_{(2)} A\| \cdot \|A\|^{j-1} d\mu \\
&\quad + c_\rho^2 \frac{c}{2} \int_{f^{-1}(B_\rho(x_1))} \|A\|^{2j-2} d\mu + c(j,k,l) \int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\leq \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c_\rho^2 c(j) \int_{f^{-1}(B_\rho(x_1))} \|A\|^{2j-2} d\mu \\
&\quad + c(j,k,l) \int_M \|A\|^{2k-j} + \|A\|^l d\mu \\
&\leq \int_{f^{-1}(B_\rho(x_1))} \|\nabla_{(2)} A\|^2 d\mu + c_\rho^2 c(j, C_{AB}) \Big(\int_{f^{-1}(B_\rho(x_1))} \|A\|^m d\mu \Big)^{\frac{2j-2}{m}} \\
&+ c(j,k,l,C_{AB}) \sup_{x \in \mathbb{R}^{n+1}} c_\rho \Big[\Big(\int_{f^{-1}(B_\rho(x))} \|A\|^m d\mu \Big)^{\frac{2k-j}{m}} + \Big(\int_{f^{-1}(B_\rho(x))} \|A\|^m d\mu \Big)^{\frac{l}{m}} \Big] \\
&\leq c_h(\delta_0^{(m)}, C_{AB}, \rho, j, k, l, n) < \infty.
\end{aligned}
$$

The other cases are simpler, and estimated as in Lemma 6, finished off using Corollary 7 as above. □

This shows that for the class of constraint functions satisfying the conditions of Theorem 3 and a small curvature condition (8), the a priori bound (A1) holds. Since we only require (A1) while (8) is true, this is enough to include constraint functions satisfying the growth condition (A2) and area bound (AB) in our main theorem.

Remark. There is an alternative approach, based also on Theorem 4, which works without the assumption (AB). However this requires monotonicity of $\int |H|$ on a ball around x_1, and does not give higher dimensional results. It is relevant to h_K flow, where we have monotonicity of $\int H$ on the entire manifold, for all time. However the essential problem is that there is no known condition which rules out the case where mean curvature is becoming more negative in one part of the manifold and more positive in another part, such that the integral over the entire manifold is non-increasing, but for any small ball the integral $\int |H|$ is increasing. Also, even if such a case is ruled out, we have no way of ensuring that the special points x_1 are in the regions of M where $\int |H|$ is monotone. What we really lack is a non-trivial condition we can impose on M_0 such that monotonicity of $\int H$ implies monotonicity

of $\int |H|$, however without the maximum principle we have not been able to achieve this. Thus h_K still presents difficulty.

4. Evolution equations for integrals of curvature.

To begin, we state the following elementary evolution equations, whose proof is standard.

Lemma 9. *For $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ evolving by (CSD) the following equations hold:*

$$\begin{aligned}\frac{\partial}{\partial t}g &= 2(\Delta H)A + 2hA,\\ \frac{\partial}{\partial t}d\mu &= (Hh + H\Delta H)d\mu,\\ \frac{\partial}{\partial t}\nu &= -\nabla\Delta H, \text{ and}\\ \frac{\partial}{\partial t}A &= -\Delta^2 A + P_3^2(A) + hA * A.\end{aligned}$$

Lemma 10. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a (CSD) flow. Then the following equation holds:*

$$\frac{\partial}{\partial t}\nabla_{(k)}A = -\Delta^2\nabla_{(k)}A + hP_2^k(A) + P_3^{k+2}(A).$$

The following is an easy consequence of the above lemma.

Corollary 11. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a (CSD) flow. Then the following equation holds:*

$$\frac{\partial}{\partial t}\|\nabla_{(k)}A\|^2 = -2\left\langle \nabla_{(k)}A, \nabla^p\Delta\nabla_p\nabla_{(k)}A\right\rangle + [hP_2^k(A) + P_3^{k+2}(A)] * \nabla_{(k)}A.$$

Using Corollary 11, we derive the following integral identity.

Corollary 12. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a (CSD) flow, and γ as in (γ). Then for any $s \geq 0$,*

$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + 2\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu = \int_M \|\nabla_{(k)}A\|^2(\partial_t\gamma^s)d\mu\\ + 2\int_M \left\langle (\nabla\gamma^s)(\nabla_{(k)}A), \Delta\nabla_{(k+1)}A\right\rangle d\mu - 2\int_M \left\langle (\nabla\gamma^s)(\nabla_{(k+1)}A), \nabla_{(k+2)}A\right\rangle d\mu\\ + \int_M \gamma^s[(P_3^{k+2}(A) + hP_2^k(A)) * \nabla_{(k)}A]d\mu.\end{aligned}$$

We now wish to use interpolation to estimate the extraneous terms from integration by parts. For $k=1$, the required inequality follows easily (for $\theta, \beta > 0$):
(9)
$$(1-\beta)\int_M \|\nabla A\|^2\gamma^{s-2}d\mu \leq \theta\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + \frac{\beta + \theta[(s-2)c_{\gamma 1}]^2}{4\beta\theta}\int_M \|A\|^2\gamma^{s-4}d\mu.$$

For $k > 1$ however we need a more powerful version of the above. Let $2 \le p < \infty$, $k \in \mathbb{N}$, $s \ge kp$, and $\theta > 0$. Then we have

(10)
$$\left(\int_M \|\nabla_{(k)}A\|^p\gamma^s d\mu\right)^{\frac{1}{p}} \le \theta\left(\int_M \|\nabla_{(k+1)}A\|^p\gamma^{s+p}d\mu\right)^{\frac{1}{p}} + c\left(\int_{[\gamma>0]} \|A\|^p\gamma^{s-kp}d\mu\right)^{\frac{1}{p}},$$

where $c = c(\theta, c_{\gamma 1}, s, p)$. This is proved, essentially, by induction on the inequality (9). Details can be found in [20]. We now estimate the equality in Corollary 12.

Proposition 13. *Let $f : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* (CSD) *flow with h satisfying* (A1) *and γ a cutoff function as in (γ). Then for a fixed $\theta > 0$ and $s \ge 2k+4$,*

$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu &+ (2-\theta)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\ &\le (c+ch)\int_M \|A\|^2\gamma^{s-4-2k}d\mu + ch\int_M \left(\nabla_{(k)}[A*A]*\nabla_{(k)}A\right)\gamma^s d\mu \\ &\quad + c\int_M \left([P_3^{k+2}(A) + P_5^k(A)]*\nabla_{(k)}A\right)\gamma^s d\mu,\end{aligned}$$

where $c = c(c_{\gamma 1}, c_{\gamma 2}, s, k, \|h\|_{\infty,[0,T)}, \theta)$.

The proof is standard, and follows by using Corollary 12 and inequality (10) to deal with the derivatives of γ, estimating the result, and absorbing.

To prove Corollary 7 we also need a version of the above estimate where we do not assume (A1). For this purpose, we state the following proposition.

Proposition 14. *Let $f : M \times [0,T) \to \mathbb{R}^3$ be a* (CSD) *flow and γ a cutoff function as in (γ). Then for a fixed $\theta > 0$ and $s \ge 2k+4$,*

$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu &+ (2-\theta)\int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\ &\le ch\int_M \left(\nabla_{(k)}[A*A]*\nabla_{(k)}A\right)\gamma^s d\mu + ch\int_M \|\nabla_{(k)}A\|^2\gamma^{s-1}d\mu \\ &\quad + c\int_M \|A\|^2\gamma^{s-4-2k}d\mu + c\int_M \left([P_3^{k+2}(A) + P_5^k(A)]*\nabla_{(k)}A\right)\gamma^s d\mu,\end{aligned}$$

where $c = c(c_{\gamma 1}, c_{\gamma 2}, s, k)$.

5. Integral estimates with small concentration of curvature.

We will first need a few Sobolev and interpolation inequalities, importantly the Michael-Simon Sobolev inequality, [29].

Theorem 15 (Michael-Simon Sobolev inequality)**.** *Let $f : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion. Then for any $u \in C_c^1(M)$ we have*

$$\left(\int_M |u|^{n/(n-1)}d\mu\right)^{(n-1)/n} \le \frac{4^{n+1}}{\omega_n^{1/n}}\int_M \|\nabla u\| + |u|\ |H|d\mu,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

The eventual goal for this section is to prove local L^∞ estimates for all derivatives of curvature. Our main tool to convert L^p bounds to L^∞ bounds is the following theorem, which is an n-dimensional analogue of Theorem 5.6 from [20]. Its proof may be found in [39].

Theorem 16. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a smooth immersed hypersurface. For $u \in C^1_c(M)$, $n < p \le \infty$, $0 \le \beta \le \infty$ and $0 < \alpha \le 1$ where $\frac{1}{\alpha} = \big(\frac{1}{n} - \frac{1}{p}\big)\beta + 1$ we have*

$$\|u\|_\infty \le c\|u\|_\beta^{1-\alpha}(\|\nabla u\|_p + \|Hu\|_p)^\alpha, \tag{11}$$

where $c = c(n, p, \beta)$.

The proof follows ideas from [21] and [20]; see also Section 6 of [24]. Due to the exponent in the Michael-Simon Sobolev inequality, it is not possible to decrease the lower bound on p, even at the expense of other parameters in the inequality. This introduces a restriction on the dimension of our immersion, and is highlighted in the following local refinement to Theorem 16.

Proposition 17. *Let $n \in \{2, 3\}$. Then for any tensor T on $f : M^n \to \mathbb{R}^{n+1}$ and γ as in (γ),*

$$\|T\|^4_{\infty,[\gamma=1]} \le c\|T\|^{4-n}_{2,[\gamma>0]}\big(\|\nabla_{(2)}T\|^n_{2,[\gamma>0]} + \|TA^2\|^n_{2,[\gamma>0]} + \|T\|^n_{2,[\gamma>0]}\big), \tag{12}$$

where $c = c(c_{\gamma 1}, n)$. Assume $T = A$, and if $n = 3$ also assume (AB). *Then there exists an $\epsilon_0 = \epsilon_0(c_{\gamma 1}, c_{\gamma 2}, n)$ such that if*

$$\|A\|^n_{n,[\gamma>0]} \le \epsilon_0$$

we have

$$\|A\|^{8n-12}_{\infty,[\gamma=1]} \le c\epsilon_0\big(\|\nabla_{(2)}A\|^{2n^2-3n}_{2,[\gamma>0]} + \epsilon_0\big), \tag{13}$$

with $c = c(c_{\gamma 1}, c_{\gamma 2}, n, \epsilon_0)$ for $n = 2$ and $c = c(c_{\gamma 1}, c_{\gamma 2}, n, \epsilon_0, C_{AB})$ for $n = 3$.

The proof is similar to that of Lemma 4.3 in [20], except for the $n = 3$ case. While the first statement follows a similar proof with minor alterations, for the second statement one needs to use the $n = 3$ version of the below multiplicative Sobolev inequality, and the area bound (AB) with Hölder's inequality.

Lemma 18. *Let γ be as in (γ). Then for an immersed surface $f : M^2 \to \mathbb{R}^3$ we have*

$$\int_M \|A\|^6\gamma^s d\mu + \int_M \|A\|^2\|\nabla A\|^2\gamma^s d\mu \le c\int_{[\gamma>0]}\|A\|^2 d\mu \int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^s d\mu$$
$$+ c(c_{\gamma 1})^4\Big(\int_{[\gamma>0]}\|A\|^2 d\mu\Big)^2,$$

and for an immersion $f : M^3 \to \mathbb{R}^4$,

$$\int_M \|A\|^6\gamma^s d\mu + \int_M \|A\|^2\|\nabla A\|^2\gamma^s d\mu \le \theta\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu$$
$$+ c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^6\big)\gamma^s d\mu + c(c_{\gamma 1})^3\big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\big),$$

where $\theta \in (0, \infty)$ and $c = c(s, \theta)$ is an absolute constant.

Proof. The first statement is Lemma 4.2 in [20]. For the second, first observe that

$$\int \|\nabla A\|^3\gamma^s d\mu \le \int_M \big(\langle A, \Delta A\rangle * \nabla A + A * \nabla A * \nabla\|\nabla A\|\big)\gamma^s d\mu$$
$$+ s\int_M \big(A * \nabla A * \nabla A * \nabla\gamma\big)\gamma^{s-1} d\mu$$

$$\begin{aligned}
&\le \frac{1}{4\theta}\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + \theta\int_M \|A\|^2\|\nabla A\|^2\gamma^s d\mu \\
&\quad + \frac{(sc_{\gamma 1})^3 4^2}{3}\int_M \|A\|^3\gamma^{2s-3}d\mu + \frac{1}{6}\int_M \|\nabla A\|^3\gamma^s d\mu \\
&\le \frac{1}{4\theta}\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + \frac{\theta^3}{3}\int_M \|A\|^6\gamma^s d\mu + \frac{(sc_{\gamma 1})^3 4^2}{3}\int_M \|A\|^3\gamma^{2s-3}d\mu \\
&\quad + \frac{5}{6}\int_M \|\nabla A\|^3\gamma^s d\mu,
\end{aligned}$$

so

$$\int \|\nabla A\|^3 d\mu \le \frac{3}{2\theta}\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + 2\theta^3\int_M \|A\|^6\gamma^s d\mu + 2(sc_{\gamma 1})^3 4^2\int_{[\gamma>0]}\|A\|^3 d\mu,$$

for any $\theta \in (0,\infty)$.

Now we use the Michael-Simon Sobolev inequality with $u = \|A\|^4\gamma^{2s/3}$ to estimate

$$\begin{aligned}
\Big(\int_M \|A\|^6\gamma^s d\mu\Big)^{\frac{2}{3}} &\le c\int_M \|A\|^3\|\nabla A\|\gamma^{\frac{2s}{3}}d\mu + c\int_M \|A\|^4\|\nabla\gamma\|\gamma^{\frac{2s-3}{3}}d\mu + c\int_M \|A\|^5\gamma^{\frac{2s}{3}}d\mu \\
&\le c\int_M \|\nabla A\|^2\|A\|\gamma^s d\mu + c\int_M \|A\|^5\gamma^s d\mu + c(c_{\gamma 1})^2\|A\|^3_{3,[\gamma>0]} \\
&\le c\int_M \|\nabla A\|^2\|A\|\gamma^s d\mu + \Big(\int_M \|A\|^6\gamma^s d\mu\Big)^{\frac{2}{3}}\Big(\int_{[\gamma>0]}\|A\|^3 d\mu\Big)^{\frac{1}{3}} \\
&\quad + c(c_{\gamma 1})^2\|A\|^3_{3,[\gamma>0]},
\end{aligned}$$

so

$$\begin{aligned}
\int_M \|A\|^6\gamma^s d\mu &\le c\Big(\int_M \|\nabla A\|^2\|A\|\gamma^s d\mu\Big)^{\frac{3}{2}} + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \|A\|^6\gamma^s d\mu \\
&\quad + c(c_{\gamma 1})^3\|A\|^{\frac{9}{2}}_{3,[\gamma>0]} \\
&\le c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^6\big)\gamma^s d\mu + c(c_{\gamma 1})^3\|A\|^{\frac{9}{2}}_{3,[\gamma>0]}.
\end{aligned}$$

This estimates the first term. For the second, we can employ a more direct technique using our estimates above,

$$\begin{aligned}
\int_M \|A\|^2\|\nabla A\|^2\gamma^s d\mu &\le c\int_M \|A\|^6\gamma^s d\mu + c\int_M \|\nabla A\|^3\gamma^s d\mu \\
&\le \theta\int_M \|\nabla_{(2)}A\|^2\gamma^s d\mu + c_\theta\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^6\big)\gamma^s d\mu \\
&\quad + c_\theta(c_{\gamma 1})^3\big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\big).
\end{aligned}$$

This estimates the second term, and combining the two estimates above finishes the proof. □

The proposition used for the constructive part of the final argument can now be proved.

Proposition 19. *Let $n \in \{2,3\}$. Suppose $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ is a* (CSD) *flow with h satisfying* (A1) *and γ a cutoff function as in* (γ). *Additionally, if $n = 3$ assume* (AB). *Then there is an $\epsilon_0 = \epsilon_0\big(c_{\gamma 1}, c_{\gamma 2}, \|h\|_{\infty,[0,T^*]}\big)$ such that if*

$$\epsilon = \sup_{[0,T^*]} \int_{[\gamma>0]} \|A\|^n d\mu \le \epsilon_0 \tag{14}$$

then for any $t \in [0,T^*]$ we have

$$\begin{aligned} \int_{[\gamma=1]} \|A\|^2 d\mu + \int_0^t \int_{[\gamma=1]} (\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6) d\mu d\tau \\ \le \int_{[\gamma>0]} \|A\|^2 d\mu\Big|_{t=0} + c\epsilon^{\frac{2}{n}} t, \end{aligned} \tag{15}$$

where $c = c\big(c_{\gamma 1}, c_{\gamma 2}, \|h\|_{\infty,[0,T^*]}, C_{AB}\big)$.

Proof. For $n = 2$, similar to [20], except for the extra integrals arising from the constraint function. These are dealt with using (A1) and absorbing. The details are similar to the $n = 3$ case, which we will describe below. Setting $k = 0$ and $s = 4$ in Proposition 13 we have

$$\begin{aligned} \frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta)\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu \le (c+ch)\int_{[\gamma>0]} \|A\|^2 d\mu \\ + ch\int_M ([A*A]*A)\,\gamma^4 d\mu + c\int_M \big([P_3^2(A) + P_5^0(A)]*A\big)\,\gamma^4 d\mu. \end{aligned} \tag{16}$$

First we estimate the P-style terms:

$$\begin{aligned} &\int_M \big([P_3^2(A) + P_5^0(A)]*A\big)\,\gamma^4 d\mu \\ &\quad\le c\int_M \big[\|A\|^3 \cdot \|\nabla_{(2)}A\| + \|\nabla A\|^2 \cdot \|A\|^2 + \|A\|^6\big]\gamma^4 d\mu \\ &\quad\le \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu + c\int_M (\|A\|^6 + \|\nabla A\|^2\|A\|^2)\gamma^4 d\mu. \end{aligned}$$

We use Lemma 18 to estimate the second integral and obtain (recall $n = 3$)

$$\begin{aligned} &\int_M \big([P_3^2(A) + P_5^0(A)]*A\big)\,\gamma^4 d\mu \\ &\quad\le \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu + c\|A\|_{3,[\gamma>0]}^{\frac{3}{2}}\int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu \\ &\qquad + c\big(c_{\gamma 1}\big)^3\big(\|A\|_{3,[\gamma>0]}^3 + \|A\|_{3,[\gamma>0]}^{\frac{9}{2}}\big). \end{aligned} \tag{17}$$

We add the integrals $\int_M \|A\|^6\gamma^4 d\mu$ and $\int_M \|\nabla A\|^2\|A\|^2\gamma^4 d\mu$ to the estimate (16) and obtain

$$\begin{aligned} &\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta)\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4 d\mu \\ &\quad\le (c+ch)\int_{[\gamma>0]} \|A\|^2 d\mu + ch\int_M ([A*A]*A)\,\gamma^4 d\mu \\ &\qquad + c\int_M \big(\|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4 d\mu + c\int_M \big([P_3^2(A) + P_5^0(A)]*A\big)\,\gamma^4 d\mu \end{aligned}$$

$$\leq c(1+h^2)\int_{[\gamma>0]}\|A\|^2 d\mu + c\int_M \big(\|A\|^3\|\nabla_{(2)}A\| + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4 d\mu.$$

We now use (17) to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta)\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4 d\mu\\
&\quad\leq c(1+h^2)\int_{[\gamma>0]}\|A\|^2 d\mu + \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu\\
&\qquad + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu\\
&\qquad + c\big(c_{\gamma 1}\big)^3\big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\big).\\
&\quad\leq c(1+h^2)C_{AB}^{\frac{1}{3}}\|A\|^2_{3,[\gamma>0]} + \theta\int_M \|\nabla_{(2)}A\|^2\gamma^4 d\mu\\
&\qquad + c\|A\|^{\frac{3}{2}}_{3,[\gamma>0]}\int_M (\|\nabla_{(2)}A\|^2 + \|A\|^6)\gamma^4 d\mu\\
&\qquad + c\big(c_{\gamma 1}\big)^3\big(\|A\|^3_{3,[\gamma>0]} + \|A\|^{\frac{9}{2}}_{3,[\gamma>0]}\big).
\end{aligned}$$

Absorbing,

$$\begin{aligned}
&\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + (2-\theta-\sqrt{\epsilon_0})\int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4 d\mu\\
&\quad\leq c\big(1 + C_{AB}^{\frac{1}{3}} + C_{AB}^{\frac{1}{3}}\|h\|^2_{\infty,[0,T^*]} + \epsilon_0^{\frac{23}{6}} + \epsilon_0^{\frac{4}{3}}\big)\epsilon^{\frac{2}{3}}\\
&\quad\leq c\epsilon^{\frac{2}{3}}.
\end{aligned}$$

For θ, ϵ_0 small enough we have

$$\frac{d}{dt}\int_M \|A\|^2\gamma^4 d\mu + \int_M \big(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6\big)\gamma^4 d\mu \leq c\epsilon^{\frac{2}{3}}.$$

Integrating,

$$\begin{aligned}
\int_{[\gamma=1]}\|A\|^2\gamma^4 d\mu &+ \int_0^t\int_{[\gamma=1]}(\|\nabla_{(2)}A\|^2 + \|A\|^2\|\nabla A\|^2 + \|A\|^6)d\mu d\tau\\
&\leq \int_{[\gamma>0]}\|A\|^2 d\mu\Big|_{t=0} + c\epsilon^{\frac{2}{3}},
\end{aligned}$$

where we used the fact $[\gamma=1]\subset[\gamma>0]$ and $0\leq\gamma\leq 1$, with

$$c = c(\epsilon_0, \|h\|_{\infty,[0,t^*]}, c_{\gamma 1}, c_{\gamma 2}, C_{AB}).$$

□

Remark. The assumption (AB) required for the three dimensional case is due to the fact that L^2 norms naturally arise when computing the evolution equations of various integral quantities, see the proof of Corollary 12 and Proposition 13. Forcing L^3 norms in these inequalities for the purpose of the above proof introduces changes in the exponents of the P-terms, and to deal with this one would need to prove an altered form of Lemma 18. This altered form will still require (AB) to handle the different exponents in the integrals. So it seems to us that for the three dimensional

case it is not possible to avoid assuming (AB), which is required to obtain results for non-trivial constraint functions regardless (see Theorem 3).

It remains only to prove the estimate used in the contradiction branch of the argument used to prove the lifespan theorem. For this, we need some interpolation inequalities, and a preliminary proposition. We will only state the required interpolation inequality; the proof can be found in [20].

Proposition 20. *Let $0 \le i_1, \ldots, i_r \le k$, $i_1 + \ldots + i_r = 2k$ and $s \ge 2k$. Then for any tensor T defined over an immersed hypersurface f we have*

$$\int_M \nabla_{(i_1)}T * \cdots * \nabla_{(i_r)}T\gamma^s d\mu \le c\|T\|^{r-2}_{\infty,[\gamma>0]}\left(\int_M \|\nabla_{(k)}T\|^2\gamma^s d\mu + \|T\|^2_{2,[\gamma>0]}\right).$$

We now use this to derive the required proposition.

Proposition 21. *Suppose $f : M^n \times [0,T] \to \mathbb{R}^{n+1}$ is a* (CSD) *flow and $\gamma : M \to \mathbb{R}$ a cutoff function as in* (γ). *Then, for $s \ge 2k+4$ the following estimate holds:*

(18)
$$\begin{aligned}\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu &+ \int_M \|\nabla_{(k+2)}A\|^2\gamma^s d\mu \\ &\le c\|A\|^4_{\infty,[\gamma>0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + c\|A\|^2_{2,[\gamma>0]}(1+\|A\|^4_{\infty,[\gamma>0]}) \\ &\quad + ch\left(h^{\frac{1}{3}}\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + (1+h^{\frac{1}{3}})\|A\|^2_{2,[\gamma>0]}\right).\end{aligned}$$

Proof. The proof is similar to [20], except for the terms which involve the constraint function. The nontrivial term is estimated as follows. Let $r = 3$ and $i_1 + i_2 = k$, $i_3 = k$ in Corollary 20 to obtain

$$\begin{aligned}h\int_M \left(\nabla_{(k)}[A*A]*\nabla_{(k)}A\right)\gamma^s d\mu &\le ch\sum_{\substack{i_1+i_2=k\\0\le i_j\le k}}\int_M \nabla_{(i_1)}A*\nabla_{(i_2)}A*\nabla_{(i_3)}A\gamma^s d\mu \\ &\le ch\|A\|_\infty\left(\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right) \\ &\le c\|A\|^4_\infty\left(\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right) \\ &\quad + h^{\frac{4}{3}}\left(\int_M \|\nabla_{(k)}A\|^2\gamma^s d\mu + \|A\|^2_{2,[\gamma>0]}\right),\end{aligned}$$

using Young's inequality. □

We now finish this section with a proof of the higher derivatives of curvature estimate, which will allow us to both bound the constraint function in balls other than the 'special ball' (see Corollary 8) and perform the contradiction part of our overall argument used to prove the lifespan theorem.

Proposition 22. *Let $n \in \{2,3\}$. Suppose $f : M^n \times [0,T^*] \to \mathbb{R}^{n+1}$ is a* (CSD) *flow with h satisfying* (A1) *and γ as in* (γ). *If $n = 3$ assume in addition* (AB). *Then there is an ϵ_0 depending on the constants in* (γ) *and $\|h\|_{\infty,[0,T^*]}$ such that if*

$$\sup_{[0,T^*]}\int_{[\gamma>0]}\|A\|^n d\mu \le \epsilon_0, \tag{19}$$

we can conclude

$$\|\nabla_{(k)}A\|^2_{\infty,[\gamma=1]} \le c\big(k, T^*, c_{\gamma 1}, c_{\gamma 2}, \|h\|_{\infty,[0,T^*]}, \alpha_0(k+2), C_{AB}\big). \tag{20}$$

Proof. The idea is to use our previous estimates and then integrate. We fix γ and consider special, tailor-made cutoff functions $\gamma_{\sigma,\tau}$ which will allow us to combine our previous estimates. Define for $0 \le \sigma < \tau \le 1$ functions $\gamma_{\sigma,\tau} = \psi_{\sigma,\tau} \circ \gamma$ satisfying $\gamma_{\sigma,\tau} = 0$ for $\gamma \le \sigma$ and $\gamma_{\sigma,\tau} = 1$ for $\gamma \ge \tau$. The function $\psi_{\sigma,\tau}$ is chosen such that $\gamma_{\sigma,\tau}$ satisfies equation (γ), although with different constants. Acceptable choices are

$$c_{\gamma_{\sigma,\tau}1} = \|\nabla\psi_{\sigma,\tau}\|_\infty \cdot c_{\gamma 1}, \text{ and } c_{\gamma_{\sigma,\tau}2} = \max\{\|\nabla_{(2)}\psi_{\sigma,\tau}\|_\infty \cdot c^2_{\gamma 1}, \|\nabla\psi_{\sigma,\tau}\|_\infty \cdot c_{\gamma 2}\}.$$

Using the cutoff function $\gamma_{0,\frac12}$ instead of γ in Proposition 19 gives

$$\int_0^{T^*}\int_{[\gamma\ge\frac12]} \|\nabla_{(2)}A\|^2 + \|A\|^6 d\mu d\tau \le c\epsilon_0(1+T^*) \tag{21}$$

for $n = 2$ and

$$\int_0^{T^*}\int_{[\gamma\ge\frac12]} \|\nabla_{(2)}A\|^2 + \|A\|^6 d\mu d\tau \le c\epsilon_0^{\frac23}(C_{AB}^{\frac13} + T^*)$$

for $n = 3$. Next, using $\gamma_{\frac12,\frac34}$ in (12) and equation (21) above we obtain for $n = 2$

$$\int_0^T \|A\|^4_{\infty,[\gamma\ge\frac34]} d\tau \le c\epsilon_0(c\epsilon_0(1+T^*) + \epsilon_0 T^*) \le c\epsilon_0. \tag{22}$$

For $n = 3$ we have

$$\int_0^T \|A\|^4_{\infty,[\gamma\ge\frac34]} d\tau \le c(C_{AB})^{\frac13}\epsilon_0^{\frac23}\Big(2[c\epsilon_0^{\frac23}(C_{AB}^{\frac13}+T^*)]^{\frac32} + c\epsilon_0(C_{AB})^{\frac12}(T^*)^{\frac32}\Big) \le c\epsilon_0, \tag{23}$$

where $c = c\big(\|h\|_\infty, c_{\gamma 1}, c_{\gamma 2}, T^*, n, \epsilon_0\big)$ for $n = 2$ and $c = c\big(\|h\|_\infty, c_{\gamma 1}, c_{\gamma 2}, T^*, n, \epsilon_0, C_{AB}\big)$ for $n = 3$. We use the convention that for the remainder of this proof all constants c will depend on these quantities for $n = 2$ and $n = 3$ respectively.

We now use (18) with $\gamma_{\frac34,\frac78}$. Factorising, we have

$$\begin{aligned}
\frac{d}{dt}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu &\le c\|A\|^4_{\infty,[\gamma_{\frac34,\frac78}\ge 0]}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu \\
&\quad + c\|A\|^2_{2,[\gamma_{\frac34,\frac78}\ge 0]}\Big(1 + \|A\|^4_{\infty,[\gamma_{\frac34,\frac78}\ge 0]}\Big) \\
&\quad + ch\Big(h^{\frac13}\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu + (1+h^{\frac13})\|A\|^2_{2,[\gamma_{\frac34,\frac78}\ge 0]}\Big) \\
&\le c\Big(\|A\|^4_{\infty,[\gamma\ge\frac34]} + h^{\frac43}\Big)\int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac34,\frac78}d\mu \\
&\quad + c\|A\|^2_{2,[\gamma\ge\frac34]}\Big(1 + \|A\|^4_{\infty,[\gamma\ge\frac34]} + h + h^{\frac43}\Big).
\end{aligned}$$

Noting that the relevant integral quantities are bounded, we apply Gronwall's inequality and obtain

$$\int_{[\gamma\ge\frac78]} \|\nabla_{(k)}A\|^2 d\mu \le \beta(t) + \int_0^t \beta(\tau)\lambda(\tau)e^{\int_\tau^t \lambda(\nu)d\nu}d\tau \le c\big(k, \alpha_0(k)\big),$$

where

$$\beta(t) = \int_M \|\nabla_{(k)}A\|^2\gamma^s_{\frac{3}{4},\frac{7}{8}}d\mu\Big|_{t=0} + c\int_0^t \Big[\|A\|^2_{2,[\gamma\geq\frac{3}{4}]}\Big(1+\|A\|^4_{\infty,[\gamma\geq\frac{3}{4}]}+h+h^{\frac{4}{3}}\Big)\Big]\,d\tau,$$
$$\lambda(t) = \|A\|^4_{\infty,[\gamma\geq\frac{3}{4}]}+h^{\frac{4}{3}}.$$

Trivially, we also have

$$\int_{[\gamma\geq\frac{7}{8}]}\|\nabla_{(k+2)}A\|^2d\mu \leq c\big(k+2,\alpha_0(k+2)\big).$$

Therefore using (13) with $\gamma_{\frac{7}{8},\frac{15}{16}}$ we can bound $\|A\|_\infty$ on a smaller ball:

$$\|A\|^{8n-12}_{\infty,[\gamma\geq\frac{15}{16}]} \leq c\epsilon_0\Big(\big[c(2,\alpha_0(2))\big]^{\frac{2n^2-3n}{2}}+\epsilon_0\Big).$$

Finally, using (12) with $T=\nabla_{(k)}A$ and $\gamma=\gamma_{\frac{15}{16},1}$ we obtain

$$\begin{aligned}\|\nabla_{(k)}A\|^4_{\infty,[\gamma=1]} &\leq c\|\nabla_{(k)}A\|^{4-n}_{2,[\gamma>\frac{15}{16}]}\Big(\|\nabla_{(k+2)}A\|^n_{2,[\gamma>\frac{15}{16}]}\\ &\qquad + (\|A\|^{2n}_{\infty,[\gamma>\frac{15}{16}]}+1)\|\nabla_{(k)}A\|^n_{2,[\gamma>\frac{15}{16}]}\Big)\\ &\leq c\big(k,\alpha_0(k+2)\big).\end{aligned}$$

This completes the proof of the proposition. □

6. Proof of the lifespan theorem.

Rescaling $\tilde{f}(x,t) = f(\frac{x}{\rho},\frac{t}{\rho^4})$, the scale invariance of

$$\int_{f^{-1}(B_\rho)}\|A\|^n d\mu$$

implies that we need only prove the theorem for $\rho = 1$. We will show that

$$\tilde{T} \geq \frac{1}{c},$$

and so scaling back we will conclude inequality (2).

We make the definition

$$\eta(t) = \sup_{x\in\mathbb{R}^{n+1}}\int_{f^{-1}(B_1(x))}\|A\|^n d\mu. \tag{24}$$

By covering B_1 with several translated copies of $B_{\frac{1}{2}}$ there is a constant c_η such that

$$\eta(t) \leq c_\eta \sup_{x\in\mathbb{R}^{n+1}}\int_{f^{-1}(B_{\frac{1}{2}}(x))}\|A\|^n d\mu. \tag{25}$$

By short time existence we have that $f(M\times[0,t])$ is compact for $t<T$ and so the function $\eta:[0,T)\to\mathbb{R}$ is continuous. We now define

(26)
$$t_0^{(n)} = \begin{cases}\sup\{0\leq t\leq \min(T,\lambda_2) : \eta(\tau)\leq 3c_\eta\epsilon_0 \ \text{ for } \ 0\leq\tau\leq t\}, & n=2,\\ \sup\{0\leq t\leq \min(T,\lambda_3) : \eta(\tau)\leq 3c_{P22}c_\eta C_{AB}^{1/3}\epsilon_0^{2/3} \ \text{ for } \ 0\leq\tau\leq t\}, & n=3,\end{cases}$$

where λ_n is a parameter to be specified later. Recall that we assume (AB) in the case where $n=3$. The constant c_{P22} is the maximum of 1 and the constant from Proposition 22 with $k=0$. Note that the ϵ_0 on the right hand side of the inequality is from equation (1).

The proof continues in three steps. First, we show that it must be the case that $t_0^{(n)} = \min(T, \lambda_n)$. Second, we show that if $t_0^{(n)} = \lambda_n$, then we can conclude the lifespan theorem. Finally, we prove by contradiction that if $T \neq \infty$, then $t_0^{(n)} \neq T$. We label these steps as

$$t_0^{(n)} = \min(T, \lambda_n), \tag{27}$$

$$t_0^{(n)} = \lambda_n \quad \Longrightarrow \quad \text{lifespan theorem}, \tag{28}$$

$$T \neq \infty \quad \Longrightarrow \quad t_0^{(n)} \neq T. \tag{29}$$

The three statements (27), (28), (29) together imply the lifespan theorem. We now give the proof of the first step, statement (27).

From the assumption (1),

$$\eta(0) \le \epsilon_0 < \begin{cases} 3c_\eta \epsilon_0, & \text{for } n = 2 \\ 3c_{P22} c_\eta C_{AB}^{1/3} \epsilon_0^{2/3}, & \text{for } n = 3, \end{cases}$$

and therefore (26) implies $t_0^{(n)} > 0$. Assume for the sake of contradiction that $t_0^{(n)} < \min(T, \lambda_n)$. Then from the definition (26) of $t_0^{(n)}$ and the continuity of η we have

$$\eta\big(t_0^{(n)}\big) = \begin{cases} 3c_\eta \epsilon_0, & \text{for } n = 2 \\ 3c_{P22} c_\eta C_{AB}^{1/3} \epsilon_0^{2/3}, & \text{for } n = 3, \end{cases} \tag{30}$$

so long as $\epsilon_0 \le 1$ and $C_{AB}, c_{P22} \ge 1$. Recall Proposition 19. We will now set γ to be a cutoff function as in (γ) such that

$$\chi_{B_{\frac{1}{2}}(x)} \le \tilde{\gamma} \le \chi_{B_1(x)},$$

for any $x \in M_t$. Choosing a small enough ϵ_0 (by varying ρ in (1)), definition (26) implies that the smallness condition (14) is satisfied on $[0, t_0^{(n)})$. Due to our assumption (A1), we also have that $\|h\|_{\infty,[0,t_0^{(n)})} < \infty$. Therefore we have satisfied all the requirements of Proposition 19 , and so we conclude

$$\begin{aligned} \int_{f^{-1}(B_{\frac{1}{2}}(x))} \|A\|^2 d\mu &\le \int_{f^{-1}(B_1(x))} \|A\|^2 d\mu \bigg|_{t=0} + c_0 c_\eta \epsilon_0^{\frac{2}{n}} t \\ &\le \begin{cases} 2\epsilon_0, & \text{for } n = 2 \text{ and } \lambda_2 = \frac{1}{c_0 c_\eta}, \\ 2c_{P22} C_{AB}^{1/3} \epsilon_0^{2/3}, & \text{for } n = 3 \text{ and } \lambda_3 = c_{P22} \frac{C_{AB}^{1/3}}{c_0 c_\eta}, \end{cases} \end{aligned} \tag{31}$$

for all $t \in [0, t^*]$, where $t^* < t_0^{(n)}$ and c_0 is the constant from Proposition 19. That is, equation (31) above is true for all $t \in \big[0, t_0^{(n)}\big)$. We combine this with (25) and Proposition 22 to conclude

$$\eta(t) \le c_{P22}^{n-2} c_\eta \sup_{x \in \mathbb{R}^{n+1}} \int_{f^{-1}(B_{\frac{1}{2}}(x))} \|A\|^2 d\mu \le \begin{cases} 2c_\eta \epsilon_0, & \text{for } n = 2 \\ 2c_{P22} c_\eta C_{AB}^{1/3} \epsilon_0^{2/3}, & \text{for } n = 3, \end{cases} \tag{32}$$

where $0 \le t < t_0^{(n)}$.

Since η is continuous, we can let $t \to t_0^{(n)}$ and obtain a contradiction with (30). Therefore, with the choice of λ_n in equation (31), the assumption that $t_0^{(n)} < \min(T, \lambda_n)$ is incorrect. Thus we have shown (27), the first of our three steps.

We in fact have also proved the second step (28). Observe that if $t_0^{(n)} = \lambda_n$ then by the definition (26) of $t_0^{(n)}$,

$$T \geq \lambda_n,$$

which is (2). Also, (32) implies (3). That is, we have proved if $t_0^{(n)} = \lambda_n$, then the lifespan theorem holds, which is the second step (28). It only remains to prove equation (29).

We assume

$$t_0^{(n)} = T \neq \infty;$$

since if $T = \infty$ then (2) holds automatically and again (32) implies (3). Note also that we can safely assume $T < \lambda_n$, since otherwise we can apply step two to conclude the lifespan theorem.

Since $T < \lambda_n$, (A1) infers the existence of a $\varsigma > 0$ such that

$$\|h\|_{\infty,[0,T+\varsigma)} < \infty,$$

which is enough (in terms of the constraint function) for short time existence to begin again at time T. To show that we may also extend the immersion f to a time interval $[0, T + \varsigma)$, we use Proposition 22 and follow a standard proof such as that found in [20] or [14]. Therefore we can extend the flow, contradicting the maximality of T.

This establishes (29) and the theorem is proved. $\square$

7. Concluding remarks.

As mentioned earlier, Kuwert & Schätzle [20] proved a lifespan theorem for the Willmore flow,

$$\frac{\partial}{\partial t} f = \big(\Delta H + Q(A)\big)\nu,$$

where they considered surfaces immersed in $\mathbb{R}^n$ via f, i.e. $f : M^2 \to \mathbb{R}^n$. Note that in one codimension $Q(A) = \|A^o\|^2 H$. We remark that one may consider the evolution equation

$$\frac{\partial}{\partial t} f = \big(\Delta H + \tilde{Q}(A)\big)\nu,$$

where $f : M^2 \to \mathbb{R}^3$, with $\tilde{Q}(A)$ a term which may be estimated as

$$\tilde{Q} \leq P_3^0(A) \tag{33}$$

and recover a lifespan theorem. One may employ exactly the techniques in [20] to obtain this result. This is essentially due to the integral estimates not depending on the precise form of the P-style terms. It may be possible to improve the growth condition (33) above to include some derivatives and more copies of A, however we have not pursued this. Of course combining this remark with the analysis we present in this paper for constrained flows will give a lifespan theorem for flows of the form

$$\frac{\partial}{\partial t} = \big(\Delta H + P_3^0(A) + h\big)\nu.$$

Apart from constrained Willlmore flows (for which one may compute constraint functions which give monotone area, volume, etc) we are not aware of any interesting examples of such flows. For immersions of dimension greater than 3, one will still be restricted by the Sobolev inequality Theorem 16, and the local version Proposition

17. We are not aware of any technique which may be used to completely remove this restriction.

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References

[1] Amann, H.: Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In: H.J. Schmeisser, H. Triebel (eds.) Function Spaces, Differential Operators and Nonlinear Analysis, pp. 9–126. Teubner, Stuttgart, Leipzig (1993)

[2] Baras, P., Duchon, J., Robert, R.: Evolution d'une interface par diffusion de surface. Comm. Partial Differential Equations **9**(4), 313–335 (1984)

[3] Burago, Y., Zalgaller, V.: Geometric Inequalities. Springer (1988)

[4] Cahn, J., Elliott, C., Novick-Cohen, A.: The Cahn-Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature. European J. Appl. Math. **7**(3), 287–301 (1996)

[5] Cahn, J., Taylor, J.: Surface motion by surface diffusion. Acta Metall. Mater. **42**(4), 1045–1063 (1994)

[6] Chruściel, P.: Semi-global existence and convergence of solutions of the Robinson-Trautman (2-dimensional Calabi) equation. Comm. Math. Phys. **137**(2), 289–313 (1991)

[7] Davì, F., Gurtin, M.: On the motion of a phase interface by surface diffusion. Z. Angew. Math. Phys. **41**(6), 782–811 (1990)

[8] Eidelman, S., Zhitarashu, N.: Parabolic Boundary Value Problems. Birkhäuser (1998)

[9] Elliott, C., Garcke, H.: Existence results for diffusive surface motion laws. Adv. Math. Sci. Appl. **7**(1), 465–488 (1997)

[10] Escher, J., Mayer, U., Simonett, G.: The surface diffusion flow for immersed hypersurfaces. SIAM J. Math. Anal. **29**(6), 1419–1433 (1998)

[11] Fife, P.: Models for phase separation and their mathematics. Electron. J. Differential Equations **2000**(48), 1–26 (2000)

[12] Friedman, A.: Partial differential equations of parabolic type. Prentice-Hall Englewood Cliffs, NJ (1964)

[13] Hamilton, R.S.: Three-manifolds with positive Ricci curvature. J. Differential Geom. **17**(2), 255–306 (1982)

[14] Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Differential Geom. **20**(1), 237–266 (1984)

[15] Huisken, G.: The volume preserving mean curvature flow. J. Reine Angew. Math. **1987**(382), 35–48 (1987)

[16] Huisken, G., Polden, A.: Geometric evolution equations for hypersurfaces. Calculus of Variations and Geometric Evolution Problems (Cetraro, 1996), Springer, Berlin pp. 45–84 (1999)

[17] Ito, K.: The surface diffusion flow equation does not preserve the convexity. RIMS Kôkyûroku Bessatsu **1105**, 10–21 (1999)

[18] Kuwert, E.: Private communication (2008)

[19] Kuwert, E., Schatzle, R.: The Willmore flow with small initial energy. J. Differential Geom. **57**(3), 409–441 (2001)
[20] Kuwert, E., Schatzle, R.: Gradient flow for the Willmore functional. Comm. Anal. Geom. **10**(2), 307–339 (2002)
[21] Ladyzhenskaya, O.A., Solonnikov, V.A., Ural'tseva, N.N.: Linear and quasi-linear equations of parabolic type. Transl. Math. Monogr. (1968)
[22] Li, S.: Quasilinear evolution problems. Ph.D. thesis, T.U. Delft (1992)
[23] Liu, Z.: Lifespan of a smooth solution for the surface diffusion flow. Chinese Ann. Math. Ser. B **24**(3), 331–342 (2003)
[24] Mantegazza, C.: Smooth geometric evolutions of hypersurfaces. Geom. Funct. Anal. **12**(1), 138–182 (2002)
[25] Mayer, U.: Numerical solutions for the surface diffusion flow in three space dimensions. J. Comput. Appl. Math. **20**(3), 361–379 (2001)
[26] McCoy, J.: The surface area preserving mean curvature flow. Asian J. Math. **7**(1), 7–30 (2003)
[27] McCoy, J.: The mixed volume preserving mean curvature flow. Math. Z. **246**(1-2), 155–166 (2004)
[28] McCoy, J.: Mixed volume preserving curvature flows. Calc. Var. Partial Differential Equations **24**(2), 131–154 (2005)
[29] Michael, J., Simon, L.: Sobolev and mean-value inequalities on generalized submanifolds of R^n. Comm. Pure Appl. Math **26**(36), 1–379 (1973)
[30] Mullins, W.: Theory of thermal grooving. J. Appl. Phys. **28**, 333–9 (1957)
[31] Polden, A.: Curves and surfaces of total curvature and fourth order flows. Ph.D. thesis, Mathematisches Institut, Uni. Tubingen (1996)
[32] Sharples, J.J.: Linear and quasilinear parabolic equations in Sobolev space. J. Differential Equations **202**(1), 111–142 (2004)
[33] Sharples, J.J.: Private communication (2008)
[34] Simon, L.: Existence of surfaces minimizing the Willmore functional. Comm. Anal. Geom **1**(2), 281–326 (1993)
[35] Simonett, G.: The Willmore flow for near spheres. Differential Integral Equations **14**(8), 1005–1014 (2001)
[36] Solonnikov, V.: On the boundary value problems for linear parabolic systems of differential equations of general form. In: Proc. Steklov Inst. Math, vol. 83, pp. 1–162 (1965)
[37] Struwe, M.: On the evolution of harmonic mappings of Riemannian surfaces. Comment. Math. Helv. **60**(1), 558–581 (1985)
[38] Topping, P.: Relating diameter and mean curvature for submanifolds of Euclidean space. Comment. Math. Helv. **83**(3), 539 (2008)
[39] Wheeler, G.: Fourth order geometric evolution equations. Ph.D. thesis, University of Wollongong (February, 2010)

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